THEORY OF PLANE FLOWS OF A READILY CONDUCTING PLASMA IN A PIPE

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Simplified equations are obtained describing slowly changing plane flows of a readily conducting quasineutral inviscid plasma in a pipe. The practically interesting case of flow in a channel with solid metal ideally conducting walls (electrodes) is analyzed. When the gas pressure is large by comparison with the magnetic pressure ($\beta \gg 1$), the field and current distribution is determined by gas dynamic factors, and the solid electrodes perturb the longitudinal electric field in a "skin" of the flow, symmetrically on the two sides of the flow, leading to attenuation of the longitudinal electric field near the input to the pipe; we also consider problems in the motion of the plasma under ideal and under poor conductivity. In the converse limiting case ($\beta \ll 1$), it is shown that as the motion of the plasma in the pipe accelerates near the anode, there is observed an increase in the intensity of the electric field which is sharply inhomogeneous in the transverse direction. The possibility of the plasma breaking away from the anode (the limiting regime) is indicated, this being accompanied by a divergence between the electron velocity and the velocity of the ions. A criterion is obtained for the breakaway of the plasma, and its possible connection with the occurrence of pre-anode "explosions" is noted. It is shown that for $\beta \ll 1$, Joule losses are small by comparison with the power in the charge and the magnitude of the losses is independent of the conductivity of the plasma.

Many theoretical and experimental papers have been devoted to the standard coaxial plasma accelerator with a special magnetic field (i.e., a field created exclusively by the electric current flowing through the accelerator); nevertheless up to this time there is no proper understanding of the processes in systems of such a kind. This can be explained both by the variety of the processes and by their complexity and their interdependence. Even if we leave on one side questions connected with the ionization of the plasma and its friction at the electrodes -- the walls of the accelerating pipe – and consider the plasma to be fully ionized and inviscid, there remains the very complex problem of the construction of the pattern of the plasma flow subject to various boundary conditions at the electrodes. The essence of this problem (which occurs for the most varied flows of a plasma near the walls) lies in the difficulty of reconciling the electromagnetic fields in the plasma flow with the fields at the electrodes* since a longitudinal electric field (cf. [1]) is necessary for the electromagnetic acceleration of a readily conducting plasma (i.e., to accelerate the ions while preserving quasineutrality). Obviously, the problem of reconciling the fields in the flow and at the electrodes does not arise if the construction of the electrodes does not impose any restrictions on the magnitude of the longitudinal electric field. Such a situation can occur in systems with sectioned electrodes [2].† However, if the electrodes restrict in some manner the possible values of the lon-

*The first indication of such a difficulty is found in [12].

†If the exchange parameter [3] is small, as is seen from the diagram of the accelerator [1], a regular pattern of the flow may be formed by going over to complete conductivity, i.e., with the aid of mass transport across the anode and the partial settling of ions on the cathode, forcing the ions to carry across the electric current.

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gitudinal electric field E_t (the simplest example is that of ideally conducting solid metal electrodes, on which $E_t = 0$), then there at once arises the problem of reconciling the fields. For this it is necessary to consider the differential equations of a higher order than the equations of motion of an ideally conducting plasma in which the inertia of the electrons is neglected, i.e., to take into account either the finite conductivity of the plasma or the inertia of the electrons or both together.

A qualitative analysis of the processes in coaxial accelerators was given in [1, 4, 5] within the framework of a two-fluid model of a plasma. The general properties of two-dimensional flows of a plasma were analyzed in [1] under the assumption that it was ideally conducting ($\sigma=\infty$) and that the electrons were inertialess ($m = 0$). It was shown that the flows have completely different properties for small exchange parameters ($\xi \ll 1$) and for $\xi \gg 1$.* The effect of the exchange parameter on the flow pattern is clarified by the fact that for cold electrodes ($T_e = 0$) such quantities as the electrical potential φ and the freezing-in characteristic B/n are preserved not along the ion trajectories but along the electron trajectories. For $\xi \ll 1$, when the ion and electron trajectories almost coincide, we can speak of the freezing in of the field "in the plasma," i.e., in both its components. But if $\xi \gg 1$, the values of B/n and φ in the volume are determined basically by their values near the cathode. Therefore, even when the conductivity is very good, but $\xi \sim 1$, a perturbation of the longitudinal electric field from the solid metal cathode is propagated throughout the whole flow. These considerations were confirmed in [4, 5], where the nature of the perturbations introduced by a weak nonideal splitting of the electrodes was analyzed (by ideally split electrodes we mean electrodes which do not impose any restrictions on E_t); these perturbations were small because of the fact that the splitting was weakly nonideal. Let us consider [4] in more detail; it was assumed in that paper that the plasma quasineutral was ($n_i \approx n_e$), the inertia of the electrons was negligibly small ($m = 0$), but that the conductivity of the plasma σ was finite, but large. The fundamental results of [4] can be formulated as follows:

1) if $\xi \ll 1$, the perturbations can be localized in symmetric skin layers near the electrodes; however, the skin thicknesses correspond to the diffusion of the plasma in the magnetic field and not to the diffusion of the field in the plasma [6]. In this case calculations using the boundary-layer method can be used;

2) but if $\xi \gg 1$, the perturbations introduced by the nonideal nature of the cathode embrace the whole volume of the pipe; this leads to the formation of a pre-anode layer. The flow can now no longer be developed from the "basic" flow and pre-electrode layers, and we must seek the solution for the whole volume of the pipe at one stroke. The strict solution of the nonlinear two-dimensional problem is extremely difficult; at the same time, the approximation of a narrow pipe clearly will not be suitable. An acceptable approach is the approximation by a pipe of slowly changing cross section developed in [1, 7] for the case of an ideally conducting plasma. This approximation is generalized here to the case of finite conductivity. In addition, we only analyze plane flows, which are the simplest, since to take account of axial symmetry is nontrivial both physically and mathematically.

It should be noted that the experimental study of flows in pipes [8, 9] completely confirms the conclusions of theory [1, 4, 5]. However, the pattern of the flows turns out to be much more complex. We are concerned with the stability of flows in coaxial accelerators. Experiments and computations on a digital computer have shown that for sufficiently large ξ, plasma flows in pipes with solid metal electrodes are unstable, resulting in the formation of pre-anode "explosions" [10, 11]. Brushlinskii, Gerlakh, and Morozov, having checked through a large number of variants, have established that there is a critical value ξ^* for given values of the magnetic Reynolds number R_m and the parameter β such that for $\xi > \xi^*$ the flow loses stability. The relation between ξ^* and the parameters R_m and β for $\beta \gg 0.1$ and given electrode geometry can be approximated by the equation [10]

$$\xi^* \sim (\beta / R_m)^{1/2}, \quad \beta = 8\pi p / B^2 \tag{0.1}$$

From this, in particular, it follows that the velocity v is restricted when there is regular outflow of the plasma. Indeed, $v \sim \xi I \leqslant \xi^* I$, and since $R_m/\beta \sim B^2 v$, $I \sim B$, v has an upper bound. Brushlinskii and Morozov investigated the stability of flows which can be described by a set of equations for two-fluid magnetic hydrodynamics, assuming ideal conductivity [12]. It appears that the flow is always unstable if there are points at which the vectors $\nabla\rho$ and $\nabla(p + B^2/8\pi)$ are not parallel. It will be shown below that the analysis of a two-fluid system of equations using the approximation of a pipe of slowly varying cross section leads to a relation similar to (0.1), but there is no comprehensive theory of pre-anode explosions at this time.

*We recall that the exchange parameter ξ (cf. [3]) is the ratio of the charge current I to the mass transport m in current units.

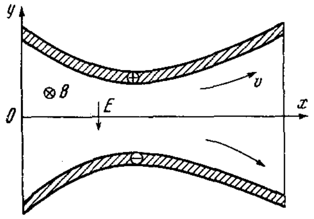

Fig. 1

1. We consider the stationary plane flow of a plasma in a transverse magnetic field, i.e., a flow for which the velocity vector **v** of the plasma, the current vector **j**, and the electric field **E** lie in the xy plane, the magnetic field **B** is perpendicular to the xy plane, and all the variables depend only on the coordinates x and y. The x axis is directed along the axis of the pipe (Fig. 1). The walls of the pipe are the electrode; they may be either solid metal or split. We shall assume that the plasma is quasineutral, completely ionized, readily conducting, inviscid, and that it does not conduct heat; we shall also assume that the ions have single charges and that the inertia of the electrons is negligibly small. We assume also that the state of the components of the plasma is described by the polytropic law. The definition of what we mean by the term "readily conducting" will be given below. The mass of an ion is denoted by M, the conductivity by σ, the density by ρ, and the gas kinetic pressures of the ionic and electronic components by p_i and p_e, respectively. Under these assumptions the equations of two-fluid magnetic hydrodynamics have the form

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla p_i}{\rho} + \frac{e}{M}\left(\mathbf{E} + \frac{v}{c}\times\mathbf{B}\right) - \frac{ej}{M\sigma} \tag{1.1}$$

$$\frac{j}{\sigma} = \mathbf{E} + \frac{v}{c}\times\mathbf{B} + \frac{M}{e\rho}\nabla p_e - \frac{M}{e\rho c}\, \boldsymbol{j}\times\boldsymbol{B} \tag{1.2}$$

$$\operatorname{div}\rho\mathbf{v} = 0, \quad \operatorname{rot}\mathbf{E} = 0, \quad \operatorname{rot}\mathbf{B} = \frac{4\pi}{c}\mathbf{j} \tag{1.3}$$

$$p_i = p_i(\rho), \;\; p_e = p_e(\rho) \tag{1.4}$$

The first two equations of (1.3) are satisfied by the introduction of the stream function Ψ (x, y) and the electric potential φ (x, y):

$$\rho v_x = \frac{\partial\Psi}{\partial y}, \;\; \rho v_y = -\frac{\partial\Psi}{\partial x}, \;\; \mathbf{E} = -\nabla\varphi \tag{1.5}$$

In addition,

$$\frac{1}{c}\,\mathbf{j}\times\mathbf{B} = -\nabla\frac{B^2}{8\pi} \tag{1.6}$$

Substituting the value of $\mathbf{j}/\sigma$ from (1.2) into (1.1) and noting (1.6), we obtain

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla\left(p_i + p_e + \frac{B^2}{8\pi}\right) \tag{1.7}$$

If we introduce the "total" plasma pressure P as the sum of the gas kinetic pressure $p = p_i + p_e$ and the magnetic pressure $B^2/8\pi$, we can write

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla P}{\rho}, \quad P = p(\rho) + \frac{B^2}{8\pi} \tag{1.8}$$

We introduce the variable μ, a typical ratio of the transverse (ionic) velocity component v_y to the longitudinal component v_x.* We shall say that a flow is slowly varying if

$$\mu \ll 1, \quad |(\mathbf{v}\cdot\nabla)v_y| \sim \mu\,|(\mathbf{v}\cdot\nabla)v_x| \tag{1.9}$$

In what follows we consider this case. We shall assume that the conductivity of the plasma is so large that

$$R_m = \frac{v_0 L}{\nu_m} \gtrsim \frac{1}{\mu^2} \gg 1 \quad \left(\nu_m = \frac{c^2}{4\pi\sigma}\right) \tag{1.10}$$

Here v_0 is a typical velocity, L a typical longitudinal scale length, ν_m the magnetic viscosity of the plasma.

Assuming that (1.9), (1.10) hold, we can significantly simplify the equations. It follows from (1.9) that $|\partial P/\partial y| \sim \mu\,|\partial P/\partial x|$, and so we can ignore the relation between P and y, assuming P to be a function only of x.

*For regimes in which the plasma does not settle at the walls, μ is a quantity of the order of the maximum absolute value of the tangent of the angle between the walls of the pipe and the x axis.

Then instead of (1.1) we can write

$$(\mathbf{v}\cdot\nabla)v_x = -\frac{1}{\rho}\frac{dP}{dx}, \quad P(x) = p(\rho) + \frac{B^2}{8\pi} \tag{1.11}$$

If (1.10) holds, we can ignore j_y/σ, the y component of the left side of (1.2). Introducing the thermalization potential φ_T by the equation

$$\varphi_T = \varphi + \frac{M}{e}\int\frac{dp_i}{\rho} \tag{1.12}$$

and noting (1.11), we can write (1.2) in its components as follows:

$$\frac{\nu_m}{c}\frac{\partial B}{\partial y} = -\frac{\partial\varphi_T}{\partial x} + \frac{v_y B}{c} + \frac{M}{e\rho}\frac{dP}{dx}, \quad 0 = \frac{\partial\varphi_T}{\partial y} + \frac{v_x B}{c} \tag{1.13}$$

Transforming to the variables x, Ψ, we obtain from (1.11)-(1.13) the following set of equations of motion:

$$\begin{gathered} v_x\frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{dP}{dx}, \quad P(x) = p(\rho) + \frac{B^2}{8\pi} \\ \frac{B}{\rho c} = -\frac{\partial\varphi_T}{\partial\Psi}, \quad \frac{\nu_m}{c}\rho v_x\frac{\partial B}{\partial\Psi} = -\frac{\partial\varphi_T}{\partial x} + \frac{M}{e\rho}\frac{dP}{dx} \end{gathered} \tag{1.14}$$

We take Eqs. (1.14) as the starting point in the subsequent discussion.* We can simplify them further if we specify the form of the function $p(\rho)$. The simplest cases are: $p \sim \rho^2$ (an adiabatic system), $p \sim \rho$ (an isothermal system); if $\beta \ll 1$, $\beta \gg 1$, the more general polytropic relation is $p \sim \rho^\gamma$. In principle, Eqs. (1.14) do not impose any restrictions on the design of the electrodes or on the conductivity as a function of the coordinates. In this paper, we consider only flows in a pipe with solid electrodes and we shall assume that the process is organized in such a way that ions are not emitted by the anode and do not settle at the cathode, i.e., the mass of the plasma passing across any transverse cross section of the pipe is constant and the electric current between the electrodes is purely electronic.

We now define the potential difference U between the anode and the cathode and the mass transport $m^{\cdot}$. Instead of the variable Ψ it is convenient to introduce the normalized coordinate $\psi = \Psi/m^{\cdot}$. Under these assumptions ψ is constant along the electrodes; we take $\psi = 0$ at the cathode and $\psi = 1$ at the anode. We further simplify equations (1.14) for the three cases.

<u>First Case.</u> Let $p = p_0(\rho/\rho_0)^\gamma$, $\beta \gg 1$. Acceleration is gas-dynamical in nature. In the zero-order approximation in β^{-1} we can assume that $\rho = \rho(x)$. The remaining equations can be written as

$$\begin{gathered} \frac{v_x^2}{2} + w(\rho) = F(\psi), \quad B = -\frac{\rho c U}{m^{\cdot}}\frac{\partial\Phi_T}{\partial\psi} \\ -\nu_m\rho^2 v_x\frac{U}{m^{\cdot 2}}\frac{\partial^2\Phi_T}{\partial\psi^2} = -U\frac{\partial\Phi_T}{\partial x} + \frac{M}{e}\frac{dw(\rho)}{dx} \\ \left(w = \int\frac{\partial p}{\rho}, \quad \Phi_T = \frac{\varphi_T}{U}\right) \end{gathered} \tag{1.15}$$

Here Φ_T is the dimensionless thermalization potential. If $F(\psi) = F_0 = \text{const}$ and $\nu_m = \nu_m(\rho)$, then $v_x = v_x(x)$. We can write the third equation of (1.15) in dimensionless form as

$$\frac{\partial^2\Phi_T}{\partial\psi^2} = \frac{\partial\Phi_T}{\partial\eta} - \frac{M}{eU}\frac{dw}{d\eta} \quad \left(\frac{d\eta}{dx} = \frac{\nu_m\rho^2 v_x}{m^{\cdot 2}}\right) \tag{1.16}$$

Differentiating (1.16) again with respect to ψ and noting the second equation of (1.15), we obtain the following equation for the freezing-in characteristic:

$$\frac{\partial^2 k}{\partial\psi^2} = \frac{\partial k}{\partial\eta} \quad \left(k = \frac{B}{\rho}\right) \tag{1.17}$$

*The equations investigated in [4] are obtained by linearizing equations (1.14) if as the unperturbed flow we take the flow in a pipe with ideally split electrodes:

$$v_x = v_x(x), \ \rho = \rho(x), \ B/\rho = \text{const}$$

Equation (1.17) does not contain a Hall term. The Hall effect enters, however, through the boundary conditions for (1.17).

Second Case. Let $p = p_0(\rho/\rho_0)^\gamma$, $\beta \ll 1$. Acceleration is electromagnetic in nature. From (1.14) in the zero-order approximation in β

$$B = B(x), \ \rho = \frac{m^* \sqrt{8\pi P(x)}}{cU\partial\Phi_T/\partial\psi} \tag{1.18}$$

The exchange parameter ξ can be written as

$$\xi = \frac{Mc}{em^*}\left(\frac{P_0}{2\pi}\right)^{1/2} \tag{1.19}$$

Here P_0 is the maximum value of P corresponding to the entrance to the pipe. The first equation of (1.14) can be written in a dimensionless form as

$$\frac{\partial u^2}{\partial q} = \frac{\partial\Phi_T}{\partial\psi} \quad \left(u = v_x\left(\frac{M}{2eU\xi}\right)^{1/2}, \ q(x) = 1 - \left(\frac{P}{P_0}\right)^{1/2}\right) \tag{1.20}$$

We transform the left side of the last equation of (1.14):

$$\frac{\nu_m}{c}\frac{\rho v_x}{m^*}\frac{\partial B}{\partial\psi} = \frac{4\pi\nu_m u}{c^2\partial\Phi_T/\partial\psi}\left(\frac{2eU\xi}{M}\right)^{1/2}\frac{\partial p}{\partial\psi} \tag{1.21}$$

$$\frac{\partial p}{\partial\psi} = \frac{\gamma p}{\rho}\frac{\partial\rho}{\partial\psi} = -\frac{\gamma p_0\rho^\gamma\partial^2\Phi_T/\partial\psi^2}{\rho_0^\gamma\,\partial\Phi_T/\partial\psi}$$

If we take as the characteristic density ρ_0 the quantity

$$\rho_0 = \frac{4\pi e\xi m^{*2}}{Mc^2U}$$

the fourth equation of (1.14) takes the form

$$u\frac{\partial^2\Phi_T}{\partial\psi^2} = \frac{c^2U^2}{4\pi\nu_m\gamma p_0}\left(\frac{M}{2eU\xi}\right)^{1/2}\frac{1}{(1-q)^\gamma}\frac{dq}{dx}\left(\frac{\partial\Phi_T}{\partial\psi}\right)^{\gamma+2}\left(\frac{\partial\Phi_T}{\partial q} + \xi\frac{\partial\Phi_T}{\partial\psi}\right) \tag{1.22}$$

The second term on the right side of (1.22) takes account of the Hall effect. As $\nu_m \to 0$, we have a regular solution for (1.22):

$$\Phi_T = f(\psi - \xi q) \tag{1.23}$$

This implies that Φ_T and with it B/ρ are preserved along the electron trajectories $\psi_e = \psi - \xi q + \text{const}$. For systems with solid metal electrodes, the solution has a different structure from (1.23) because of the effect of the boundary conditions. This question is considered in detail in §3 below.

Third Case. Let $\gamma = 2$ (an adiabatic system). In this case β may be arbitrary. Equations (1.14) can be transformed to the form

$$\frac{\partial u^2}{\partial q} = \left[\left(\frac{\partial\Phi_T}{\partial\psi}\right)^2 + \varkappa^2\right]^{1/2}, \ B = -\frac{\partial\Phi_T}{\partial\psi}\left[\frac{8\pi P}{(\Phi_T/\partial\psi)^2 + \varkappa^2}\right]^{1/2}$$

$$\rho = \frac{m^*}{cU}\left[\frac{8\pi P}{(\partial\Phi_T/\partial\psi)^2 + \varkappa^2}\right]^{1/2}\left(\varkappa = \left(\frac{p_0}{P_0}\right)^{1/2} = \beta_0^{1/2}\right)$$

$$u\frac{\partial^2\Phi_T}{\partial\psi^2} = \frac{1}{\nu_m(1-q)^2}\frac{dq}{dx}\left(\frac{M}{2eU\xi}\right)^{1/2}\left(\frac{m^*}{\rho_0\varkappa}\right)^2\left[\left(\frac{\partial\Phi_T}{\partial\psi}\right)^2 + \varkappa^2\right]^2\left\{\frac{\partial\Phi_T}{\partial q} + \xi\left[\left(\frac{\partial\Phi_T}{\partial\psi}\right)^2 + \varkappa^2\right]^{1/2}\right\} \tag{1.24}$$

We consider the first and second cases in more detail.

2. Consider the thermal regime of accelerating the plasma ($\beta \to \infty$). For this regime, with $F(\psi) = F_0 = \text{const}$, the equations of motion and the field equations have the following forms:

a) the zero-order approximation in β^{-1}:

$$\frac{v_x^2}{2} + w(\rho) = F_0, \ B = -\rho(\eta)\frac{cU}{m^*}\frac{\partial\Phi_T}{\partial\psi} \tag{2.1}$$

$$\frac{\partial^2\Phi_T}{\partial\psi^2} = \frac{\partial\Phi_T}{\partial\eta} - \frac{M}{eU}\frac{dw}{d\eta}\left(w = \int\frac{dp}{\rho}, \ \eta = \int_0^x \nu_m\frac{\rho^2 v_x}{m^{*2}}dx\right)$$

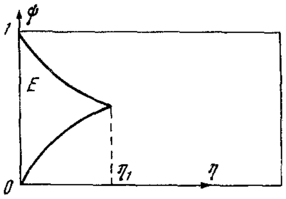

Fig. 2

We see that the dynamical and electrical problems are independent here: if we specify ρ, v_x, w, we can then find Φ_T and B. This is reasonable since in this regime the field is completely defined by the gas-dynamic nature of the flow;

b) first approximation. We denote the unperturbed values by the subscript 0 and the perturbed values by the suscript 1. Then we obtain

$$\rho_{(1)} = -\frac{B_{(0)}^2}{8\pi c_{T(0)}^2}, \quad v_{(0)}v_{(1)} = -\int \frac{\rho_{(1)}}{\rho_{(0)}} dw_{(0)} \quad \left(c_{T(0)}^2 = \frac{dp_{(0)}}{d\rho_{(0)}}\right)$$

$$\frac{\partial^2}{\partial\psi^2}\frac{B_{(1)}}{c\rho_{(0)}} = \frac{\partial}{\partial\eta}\frac{B_{(1)}}{c\rho_{(0)}} - \frac{\partial}{\partial\psi}\left[\left(\frac{\rho_{(1)}}{\rho_{(0)}} + \frac{v_{(1)}}{v_{(0)}}\right)\frac{\partial}{\partial\psi}\frac{B_{(0)}}{c\rho_{(0)}}\right] - \frac{\partial}{\partial\eta}\left(\frac{B_{(0)}}{c\rho_{(0)}}\frac{\rho_{(1)}}{\rho_{(0)}}\right) - \frac{M}{em^{\cdot}}\frac{dw_{(0)}}{d\eta}\frac{\partial}{\partial\psi}\frac{\rho_{(1)}}{\rho_{(0)}} \tag{2.2}$$

The last equation of (2.2) is mathematically equivalent to the equation of heat conduction in which the right side is a function of η and ψ. When $F = F(\psi)$ the equation is more complicated, but its structure remains as before. Returning to the original set of equations (1.1)-(1.4), it is easy to see that as $\beta \to \infty$, we can ignore the term $\mathbf{j} \times \mathbf{B}/enc$ in equation (1.2). Indeed, the order of the term $j \sim cB/4\pi L$ and the ratio of $\mathbf{j} \times \mathbf{B}/enc$ to the Lorentz term $\mathbf{v} \times \mathbf{B}/c$ are each ($v \sim c_T$):

$$\frac{cB}{4\pi Lenc_T} = \frac{c_A}{c_T L \Pi_i^{1/2}} \left(\Pi_i = \frac{4\pi e^2 n}{Mc^2}, \; c_A^2 = \frac{B^2}{4\pi\rho}\right) \tag{2.3}$$

Keeping Π_i constant and increasing $\beta \sim c^2_T/c^2_A$, we see that the Hall current is indeed negligible. Hence, on the right side of the equation for Φ_T in (2.1) the Hall term depends only on the longitudinal coordinate η, and the Hall term is absent from the equation for B/ρ in (1.17).

The simplest form of the system (2.1) can be analyzed for the case of cold ions ($p_i \to 0$). Here $\Phi_T = \Phi$, and from (2.1) we obtain

$$\frac{\partial^2\Phi}{\partial\psi^2} = \frac{\partial\Phi}{\partial\eta} - \frac{M}{eU}\frac{dw}{d\eta}, \quad 0 \leqslant \psi \leqslant 1, \quad 0 \leqslant \eta \leqslant \eta_0 \tag{2.4}$$

The boundary conditions for (2.4) have the form

$$\Phi(\eta, 0) = 0, \; \Phi(\eta, 1) = 1, \; \Phi(0, \psi) = \Phi_0(\psi) \tag{2.5}$$

Consider in more detail the case when

$$w = w_0 + \eta w_1 \; (w_1 = \text{const}, \; w_0 = \text{const})$$

Then the solution of (2.5), while preserving its general intrinsic structure, becomes somewhat simpler.* If we introduce the function Γ through the equation

$$\Gamma = \Phi - \psi + \frac{M}{2eU} w_1 (\psi^2 - \psi) \tag{2.6}$$

we reduce the problem (2.4), (2.5) to the following:

$$\frac{\partial^2\Gamma}{\partial\psi^2} = \frac{\partial\Gamma}{\partial\eta}, \; \Gamma(\eta, 0) = \Gamma(\eta, 1) = 0, \; \Gamma(0, \psi) = \Gamma_0(\psi) \tag{2.7}$$

The solution of Eq. (2.7) has the form

$$\Gamma(\eta, \psi) = 2\sum_{n=1}^{\infty} \exp[-\pi^2 n^2 \eta] \sin \pi n\psi \int_0^1 \Gamma_0(\zeta) \sin \pi n\zeta d\zeta \tag{2.8}$$

It follows from (2.8) that $\Gamma \to 0$ for $\eta > \pi^{-2}$, and so

$$\Phi \to \psi - \frac{M}{2eU} w_1 (\psi^2 - \psi) = \Phi(\psi) \quad (\eta > \pi^{-2} = \eta_1) \tag{2.9}$$

*Equation (2.4) with the boundary conditions (2.5) replaced by $\Phi = \chi + \psi + M[w(\eta) - w(0)]/eU$ leads to a problem which is easily solved by Laplace's method.

Thus, the effect of the conditions at the entry side are significant for $0 \leq \eta \lessapprox \eta_1$, and it is just in this interval that there is a longitudinal electric field; for $\eta > \eta_1$ the longitudinal electric field is exponentially small. We can say that the perturbation of the initial longitudinal electric field by the boundary conditions (in this case, at the solid electrodes) penetrates the skin into the depth of the plasma flow. In this case the pre-electrode skin layer is symmetric. The situation here is the same as when a boundary layer occurs in ordinary gas-dynamics. For $\eta > \eta_1$ the skin layers overlap, which results in the longitudinal electric field vanishing (the domain E in Fig. 2).*

In conclusion, we note two exactly solvable limiting cases. For this we need not assume that the flow is slowly varying, and the restriction on the magnetic Reynolds number can be lifted.

First Case. Let $\sigma \to \infty$. Then the equation for the freezing in of the electron component is valid [1]:

$$\frac{B}{\rho} = -c\frac{d\varphi_T}{d\psi_e}(\psi_e), \quad B = \frac{4\pi eM}{cm^*}(\psi - \psi_e) \tag{2.10}$$

Determining ρ and ψ from the equations of gas-dynamics and noting that $\varphi_T(\psi_e)$ is an a priori given function, we obtain a single equation for B. Obviously, however, in this case we have to ignore boundary effects.

Second Case. Let $\sigma \to 0$. Then Ohm's law (1.2) takes the form

$$\frac{\mathbf{j}}{\sigma} = \mathbf{E} + \frac{M}{e\rho}\nabla p_e, \quad \mathbf{E} = -\nabla\varphi \tag{2.11}$$

Assuming that $p_e = p_e(\rho)$ and noting that div $\mathbf{j} = 0$, we find that

$$\operatorname{div} \sigma\left(\frac{M}{e\rho}\nabla p_e(\rho) - \nabla\varphi\right) = 0 \tag{2.12}$$

The density ρ is defined from the gas-dynamic problem, and so Eq. (2.12) completely defines φ when the boundary conditions on φ are given; hence the distribution of the electric current is determined.

3. Consider the electrodynamic regime of accelerating the plasma ($\beta \to 0$). For flows with $\beta \to \infty$, it was shown that when the conductivity is finite, $\Phi \to \Phi(\psi)$, and thus, along a given stream line $\psi =$ const the potential is conserved. Obviously (cf. footnote on p. 534) if the pipe is sufficiently long and the conductivity finite, a regime is always established which is close to that of a flow with $\Phi = \Phi(\psi)$ (we call the latter a quasi-isomagnetic flow). For a quasi-isomagnetic flow the electric field is perpendicular to the ionic trajectory ($\mathbf{E} \cdot \mathbf{v} = 0$), and so, as we see from (1.1), the acceleration of ions in regimes with $\beta \ll 1$ is determined by the term j_x/σ, i.e., the "electron wind" [8, 13]. This regime is impactive (faster electrons, striking the ions, accelerate them) and it is naturally called ohmic. The qualitative features of a quasi-isomagnetic ohmic regime are most simply analyzed by means of the example of a flow in a narrow pipe with solid walls (the electrodes). Suppose the ions are cold, i.e., $p(\rho) = p_e(\rho)$. It follows from (1.2) that $E = vB/c$; since $U = Ef =$ const (f is the width of the pipe), the back emf is constant, $vBf/c =$ const. Since the mass flow $m^\cdot = \rho v f$ is also constant, it is thus obvious that the condition for an isomagnetic regime $B/\rho =$ const holds for the narrow pipe. As a result, we obtain from (1.14)

$$\frac{B}{\rho} = \text{const}, \quad \rho v f = \text{const}, \quad \frac{v^2}{2} + w(\rho) + \frac{B^2}{4\pi\rho} = \text{const} \quad \frac{j_x}{\sigma} = \frac{M}{e\rho}\frac{d}{dx}\left(p + \frac{B^2}{8\pi}\right) \tag{3.1}$$

The first three equations of (3.1) are analogous to the equations for the flow of an ideal plasma in a narrow pipe. In particular, it follows from this that the pipe must have a throat. The fourth equation is independent; it determines the longitudinal current in the pipe. For $\beta \ll 1$, $j_x = -\omega\tau j_y$, where $\omega\tau = M\sigma B/e\rho c$ is the Hall parameter.

The system (3.1), strictly speaking, can only be applied to a narrow "tubular current." In a pipe of finite width we have to take into account two-dimensional effects due to the presence of a longitudinal current. The quasi-isomagnetic flow

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*For cold ions and small β the situation is analogous: the longitudinal electric field dies out with distance from the entry to the pipe since the equation for Φ is of quasiparabolic type:

$$\frac{\partial^2\Phi}{\partial\psi^2} = \alpha\left(q, \frac{\partial\Phi}{\partial\psi}\right)\frac{\partial\Phi}{\partial q} + \alpha_1\left(q, \frac{\partial\Phi}{\partial\psi}\right)(\alpha > 0)$$

$$\frac{B}{\rho} = -\frac{cU}{m^{\cdot}}\frac{d\Phi(\psi)}{d\psi}$$

is a direct generalization of (3.1) to the case of a pipe of finite width.

Consider the quasi-isomagnetic regime under the assumption, for simplicity, that the conductivity of the plasma is constant.* For $\beta \ll 1$ we can assume that $\Phi_T = \Phi$ in (1.22). Taking note of (1.20), for a quasi-isomagnetic regime we have

$$u = \left[(q + q_0)\frac{d\Phi}{d\psi}\right]^{1/2} \quad (q_0 = \text{const} > 0)$$

$$\frac{d^2\Phi}{d\psi^2} = \frac{c^2U^2}{4\pi\nu_m}\left[\frac{M}{2eU\xi(q+q_0)}\right]^{1/2}\frac{\xi}{\gamma p_0(1-q)^{\gamma}}\frac{dq}{dx}\left(\frac{d\Phi}{d\varphi}\right)^{\gamma+3/2} \tag{3.2}$$

The function q(x) is determined from the condition that the factor in front of the term $\partial\Phi/\partial\psi$ on the right side of the second equation of (3.2) is a constant. Introducing a characteristic longitudinal scale length L for the decrease in the magnetic field (in an accelerating regime $dq/dx > 0$), we obtain an equation for q(x):

$$\frac{L}{(q+q_0)^{1/2}(1-q)^{\gamma}}\frac{dq}{dx} = 1 \tag{3.3}$$

If we take for the typical velocity v_0 the quantity $(2eU\xi/M)^{1/2}$ and for the typical magnetic field B_0 the quantity $cU\rho_0/m^{\cdot}$, we can transform the factor on the right side of the second equation of (3.2) to the form

$$\frac{c^2U^2\xi}{4\pi\nu_m\gamma p_0 L}\left(\frac{M}{2eU\xi}\right)^{1/2} = \frac{1}{2L}\frac{v_0^2}{c_T^2}\left(\frac{m^{\cdot}}{\rho_0 v_0}\right)(\omega\tau)_0 = a \tag{3.4}$$

Here $c_T^2 = \gamma p_0/\rho_0$; $(\omega\tau)_0$ is the typical Hall parameter. Let ε denote the value of $\partial\Phi/\partial\psi$ at the anode ($\psi = 1$). Integrating (3.2) and noting that $\Phi(0) = 0$, we obtain

$$\frac{d\Phi}{d\psi} = \frac{\varepsilon}{[1 + sa\varepsilon^s(1-\psi)]^{1/s}},$$

$$\Phi = \frac{\varepsilon}{sa\varepsilon^s}\frac{s}{s-1}\{[1 + sa\varepsilon^s]^{1-1/s} - [1 + sa\varepsilon^s(1-\psi)]^{1-1/s}\} \tag{3.5}$$

Here $s = \gamma + 3/2$ so that in fact $s > 1$. The condition $\Phi(1) = 1$ yields a relation between a and ε:

$$1 = \frac{\varepsilon}{sa\varepsilon^s}\frac{s}{s-1}\{[1 + sa\varepsilon^s]^{1-1/s} - 1\} \tag{3.6}$$

It follows from (3.6) that $\varepsilon \to 1$ as $a \to 0$, for $a > 0$, $\varepsilon > 1$. It is easy to see that the case $\varepsilon \to \infty$ is possible; this is called the limiting regime. It happens when

$$a = a^* = \frac{1}{s}\left(\frac{s}{s-1}\right)^s = \frac{2}{3+2\gamma}\left(\frac{3+2\gamma}{1+2\gamma}\right)^{\gamma+3/2} \tag{3.7}$$

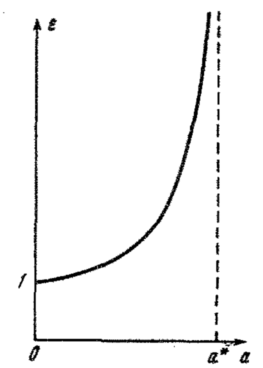

Fig. 3

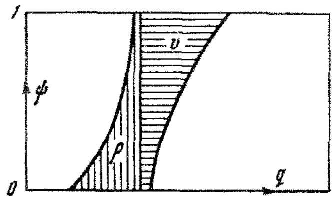

Fig. 4

*If σ depends on the temperature T, for example, $\sigma = \sigma_0(T/T_0)^{3/2}$, this case can also easily be considered, assuming a polytropic law for $p(\rho)$.

For $\gamma = 5/3$, $a^* = 1.05$. A qualitative representation of the function $\varepsilon(a)$ is shown in Fig. 3. The singularity of the solution (3.5) is a branch point. It limits the maximum possible value of ψ. Indeed, it follows formally from (3.5) that

$$\psi < 1 + \frac{1}{sa\varepsilon^3} = \psi^* \tag{3.8}$$

As $\varepsilon \to \infty$ we have $a \to a^*$ and $\psi^* \to 1$, i.e., the singularity of the solution occurs at the anode of the accelerator. The solution is regular for $0 \le a < a^*$. Using (3.4), we can write the condition that the solution be regular as

$$L \frac{c_T^2}{v_0^2} \frac{1}{(\omega\tau)_0} > \frac{1}{2a^*} \left(\frac{m^\cdot}{\rho_0 v_0}\right) \tag{3.9}$$

It follows from (1.8) and (3.2) that

$$\rho \sim \left(\frac{d\Phi}{d\psi}\right)^{-1}, \quad v_x \sim \left(\frac{d\Phi}{d\psi}\right)^{1/2}$$

A qualitative profile of ρ and v_x is shown in Fig. 4. We have $\rho \to 0$ at the anode as $d\Phi/d\Psi \to \infty$.

The longitudinal electric current presses the plasma against the anode; since the electrons move in the transverse direction due to collisions with ions, the motion is similar to that of diffusion. At the cathode the density is higher, and so it is easier for the electrons to leave the cathode than for them to reach the anode.

We see [cf. (3.5)] that $d\Phi/d\psi$ increases as ψ increases, i.e., in the direction of the anode, so that it is only near the anode that a zone of the electric field of high intensity is formed, i.e., the pre-anode layer.

Indeed, the formation of the "pre-anode layer" is connected simply with the increase in resistance as $\rho \to 0$ and not with the transfer of potential as in [4].

Formally, as the limiting regime is attained there should be a breakaway of the plasma from the anode, which would lead to a rapid redistribution of the current density in the pipe and to a collapse of the stationary regime. In fact, however, stability must be lost somewhat earlier because of the large divergence between the electron velocity and the ion velocity (it is easy to see that $v_{ex} \gg v_x \to \infty$ as $d\Phi/d\psi \to \infty$); in addition, the effect of the gas kinetic viscosity of the plasma must also affect the stability of the flow. Plasma flows in which breakaway occurs were studied in [14]. Attention is directed to the fact that the criterion (3.9) is closely similar to the condition (0.1) for the occurrence of pre-anode explosions. Indeed, if we introduce the "local" exchange parameter ξ_1 as the typical ratio of the transverse electron velocity to the longitudinal plasma velocity, we find from (3.9) that

$$\xi_1 = \frac{McB_0}{4\pi e L \rho_0 v_0} = \frac{\xi}{L}\left(\frac{m^\cdot}{\rho_0 v_0}\right), \quad \xi_1 \leqslant \left(\frac{a^* \gamma \xi}{2} \frac{\beta_0}{R_m}\right)^{1/2} \tag{3.10}$$

$$\left(R_m = \frac{v_0 L}{\nu_m}, \quad \beta_0 = 8\pi \frac{p_0}{B_0^2}\right)$$

In conclusion, we estimate the energy dissipation due to Joule losses. In regimes close to the limiting regime $|j_x| \gg |j_y|$, so that the volume is equal to j_x^2/σ. The total liberation of Joule heat in unit time is

$$Q = \int_V \frac{j_x^2}{\sigma} dx dy = \int_0^1 \sigma (U\xi)^2 \left(\frac{d\Phi}{d\psi}\right)^{5/2} \frac{m^\cdot}{\rho_0 v_0 (q+q_0)^{1/2} (1-q)} \frac{dq}{\partial x} dq d\psi \tag{3.11}$$

Integrating, we obtain the ratio of the Joule losses to the applied power N:

$$\frac{Q}{N} = \frac{\xi\beta_0}{2(\gamma-1)}\left(\frac{3+2\gamma}{1+2\gamma}\right)^{\gamma-1}, \quad \gamma > 1 \left(N = IU = \frac{em^\cdot \xi U}{M}\right) \tag{3.12}$$

$$\frac{Q}{N} = 0.2\xi\beta_0 \ln[1 + 0.4a\varepsilon^{5/2}] \approx \frac{\xi\beta_0}{2} \ln \frac{5}{3}\varepsilon, \quad \gamma = 1$$

It follows from this that the Joule losses are small for $\xi \lesssim 1$, $\beta \ll 1$. In addition, the size of the Joule losses is independent of the conductivity of the plasma.

LITERATURE CITED

1. A. I. Morozov and L. S. Solov'ev, "Plane flows of an ideally conducting compressible fluid taking account of the Hall effect," Zh. Tekh. Fiz., 34, No. 7 (1964).
2. G. Wood and A. Carter, "NASA considerations on the design of a continuous plasma accelerator with constant magnetic field," in: Ion, Plasma, and Arc Rocket Motors [in Russian], Atomizdat, Moscow (1961).
3. A. I. Morozov and L. S. Solov'ev, "A similarity parameter in the theory of plasma flows," Dokl. Akad. Nauk SSSR, 164, No. 1 (1965).
4. A. I. Morozov and A. P. Shubin, "The flow of a plasma between electrodes with weak longitudinal conductivity," Teplofiz. Vys. Temp., 3, No. 6 (1965).
5. A. I. Morozov and A. P. Shubin, "Pre-electrode layers in the flows of a readily conducting inviscid plasma," Prikl. Mekh. i Tekh. Fiz., No. 5 (1967).
6. L. Spitzer, Jr., Physics of a Fully Ionized Gas, Interscience (1962).
7. A. I. Morozov and L. S. Solov'ev, "Symmetric flows of a conducting fluid across a magnetic field," Dokl. Akad. Nauk SSSR, 154, No. 2 (1964).
8. A. Ya. Kislov, A. I. Morozov, and G. N. Tilinin, "The potential distribution in a coaxial quasistationary plasma injector," Zh. Tekh. Fiz., 38, No. 6 (1968).
9. P. E. Kovrov, A. I. Morozov, L. G. Tokarev, and G. Ya. Shchepkin, "The distribution of the magnetic field in a coaxial plasma injector," Dokl. Akad. Nauk SSSR, 172, No. 6 (1967).
10. A. I. Morozov, K. V. Brushlinskii, N. I. Gerlakh, and A. P. Shubin, "The theoretical and numerical analysis of physical processes in a stationary high-current gas discharge between coaxial electrodes," Proc. Internat. Conf. on Phenomena in Ionized Gases, Vienna, 1967, Contributed papers, Paper No. 159 (1968).
11. K. V. Brushlinskii, N. I. Gerlakh, and A. I. Morozov, "The calculation of two-dimensional stationary flows of a plasma of finite conductivity when the Hall effect is present," Magnitnaya Gidrodinamika, No. 1 (1967).
12. K. V. Brushlinskii and A. I. Morozov, "The evolution of the equations of magnetic hydrodynamics to take the Hall effect into account," Priklad. Matem. i Mekhan., 32, No. 5 (1968).
13. A. I. Morozov, E. V. Artyushkov, L. S. Solov'ev, and A.P. Shubin, "Some properties of the flow of a conducting gas in a magnetic field," in: Low-temperature Plasma [in Russian], Mir, Moscow (1967).
14. G. M. Bam-Zelikovich, "The effect of the Hall currents on the flow of a conducting gas at large velocities," Prikl. Mekh. i Tekh. Fiz., No. 3 (1965).